

MAS 117

Introduction to Algebra

Solutions 1

January 2007

1 If we write the sum as

$$1 + 2 + 3 + \cdots + (n-1) + n,$$

the i th number is obviously i . If we write it in reverse as

$$n + (n-1) + (n-2) + \cdots + 2 + 1,$$

the i th number is got by starting with n and subtracting one $i-1$ times, yielding $n - (i-1) = n - i + 1$. So the i th numbers in the two sums add up to $i + (n - i + 1) = n + 1$. Thus, we get

$$2S = (n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1) = n(n+1),$$

and so $S = n(n+1)/2$.

2 $p = 11$ is a counterexample since $2^{11} - 1 = 2047 = 23 \times 89$. This can be found by trial and error: $2^2 - 1 = 3$, $2^3 - 1 = 7$, $2^5 - 1 = 31$ and $2^7 - 1 = 127$ are all primes.

3 The mistake in the argument is confusing “if” and “only if”. Pythagoras says that “If the triangle is right-angled then $a^2 + b^2 = c^2$ ”, but we have used the converse statement “If $a^2 + b^2 = c^2$ then the triangle is right-angled” in the supposed proof.

4 (a) 1, 2, 4, 8, 16, 32, ...

(b) $a_n = 2^{n-1}$.

(c) This formula is correct for $n = 1$, since $a_1 = 1 = 2^0$. For the inductive step, suppose that $a_{n-1} = 2^{n-2}$; then

$$a_n = 2a_{n-1} = 2 \times 2^{n-2} = 2^{n-1},$$

as required. So $a_n = 2^{n-1}$ holds for all n by induction.

5 The argument is (obviously) not valid: I can clearly find two horses with different colours, so the inductive step from 1 to 2 must fail. Look at that step. For $n = 2$, we have $n - 1 = 1$, so the purported argument says

- H_1 has the same colour (as itself), and
- H_2 has the same colour (as itself).

We can't conclude that H_1 and H_2 have the same colour. The argument given implicitly assumes that the sets $\{H_1, \dots, H_{n-1}\}$ and $\{H_2, \dots, H_n\}$ overlap; but for $n = 2$ they don't.